On heat kernel upper bounds for symmetric Markov semigroups

Jian Wang (Fujian Normal University)

16th Workshop on Markov Processes and Related Topics

Central South University

July 16, 2021

Nash-type inequalities (Coulhon, 96)

Theorem 1

For a strictly decreasing differentiable bijection φ *of* $\mathbb{R}_+$ *satisfying condition* (D), *define* $\theta(r) = -\varphi'(\varphi^{-1}(r))$ *for all* $r > 0$. *Let* δ *be a non-negative constant. Then the following conditions are equivalent:*

(i) *There is a constant* $c_1 > 0$ *such that*

$$\|P_t\|_{1\to\infty} \le \varphi(c_1 t)e^{\delta t} \quad \text{for } t > 0.$$

(ii) *There is a constant* $c_2 > 0$ *such that*

$$c_2\,\theta(\|f\|_2^2) \le \mathcal{E}(f,f) + \delta\|f\|_2^2 \qquad \text{for } f \in \mathcal{F} \text{ with } \|f\|_1 \le 1.$$

For instance, C^1-functions $f(t)$ that behave like $t^{-\delta_1}$ for small t with $\delta_1 > 0$, and $e^{-c_0 t^{\delta_2}}$ for large t with $c_0, \delta_2 > 0$ satisfy condition (D).

Nash-type inequalities (Coulhon, 96)

Theorem 1

For a strictly decreasing differentiable bijection φ of $\mathbb{R}_+$ satisfying condition (D)*, define* $\theta(r) = -\varphi'(\varphi^{-1}(r))$ *for all* $r > 0$*. Let* δ *be a non-negative constant. Then the following conditions are equivalent:*

(i) *There is a constant* $c_1 > 0$ *such that*

$$\|P_t\|_{1\to\infty} \le \varphi(c_1 t)e^{\delta t} \quad \text{for } t > 0.$$

(ii) *There is a constant* $c_2 > 0$ *such that*

$$c_2\,\theta(\|f\|_2^2) \le \mathcal{E}(f,f) + \delta\|f\|_2^2 \qquad \text{for } f \in \mathcal{F} \text{ with } \|f\|_1 \le 1.$$

For instance, C^1-functions $f(t)$ that behave like $t^{-\delta_1}$ for small t with $\delta_1 > 0$, and $e^{-c_0 t^{\delta_2}}$ for large t with $c_0, \delta_2 > 0$ satisfy condition (D).

Functional inequalities (Davies, Coulhon, Wang)

- (Coulhon, 96) Nash-type inequalities

$$\theta(\|f\|_2^2) \le \mathcal{E}(f,f) + \delta\|f\|_2^2 \quad \text{for } f \in \mathcal{F} \text{ with } \|f\|_1 \le 1$$

with $\int^{+\infty} 1/\theta(s)\, ds < \infty$.

- (F.-Y. Wang, 00) super-Poincaré inequalities

$$\|f\|_2^2 \le r\mathcal{E}(f,f) + \beta(r)\|f\|_1^2, \quad f \in \mathcal{F}, r > 0.$$

- (Davies, 87) super-logarithmic Sobolev inequality

$$\int f^2 \log \frac{f^2}{\|f\|_2^2}\, dm \le r\mathcal{E}(f,f) + (\log \beta(r))\|f\|_2^2, \quad f \in \mathcal{F}, r > 0.$$

• E.B. Davies. *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge, UK, 1989.

Functional inequalities (Davies, Coulhon, Wang)

- (Coulhon, 96) Nash-type inequalities

$$\theta(\|f\|_2^2) \le \mathcal{E}(f,f) + \delta\|f\|_2^2 \quad \text{for } f \in \mathcal{F} \text{ with } \|f\|_1 \le 1$$

with $\int^{+\infty} 1/\theta(s)\, ds < \infty$.

- (F.-Y. Wang, 00) super-Poincaré inequalities

$$\|f\|_2^2 \le r\mathcal{E}(f,f) + \beta(r)\|f\|_1^2, \quad f \in \mathcal{F}, r > 0.$$

- (Davies, 87) super-logarithmic Sobolev inequality

$$\int f^2 \log \frac{f^2}{\|f\|_2^2}\, dm \le r\mathcal{E}(f,f) + (\log\beta(r))\|f\|_2^2, \quad f \in \mathcal{F}, r > 0.$$

- E.B. Davies. *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge, UK, 1989.

Nash inequalities and off-diagonal heat kernel estimates (Carlen-Kusuoka-Stroock, 87)

Theorem 2

Let $\nu > 0$ and $\delta \geq 0$. The following statements are equivalent.

(i)
$$\|f\|_2^{2+4/\nu} \leq A(\mathcal{E}(f,f) + \delta\|f\|_2^2)\|f\|_1^{4/\nu} \qquad \text{for } f \in \mathcal{F}.$$

(ii) *There is a constant $C_\nu > 0$ such that for all $\varepsilon \in (0,1)$, for all $t > 0$ and $x, y \in E \setminus \mathcal{N}$,*

$$p(t,x,y) \leq C_\nu (A/(\varepsilon t))^{\nu/2} e^{\varepsilon\delta t} \exp\left(-|\psi(y) - \psi(x)| + (1+\varepsilon)\Lambda(\psi)^2 t\right),$$

where

$$\Lambda(\psi)^2 := \max\left\{\left\|\frac{de^{-2\psi}\Gamma(e^\psi, e^\psi)}{dm}\right\|_\infty, \left\|\frac{de^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})}{dm}\right\|_\infty\right\} < \infty.$$

Nash-type inequalities and off-diagonal heat kernel estimates (Chen-Kim-Kumagai-W., 21)

Theorem 3

Let $\varphi \in \mathcal{R}$ and $\delta \geq 0$. The following statements are equivalent.

(i)
$$c_1\theta(c_2\|f\|_2^2) \leq \mathcal{E}(f,f) + \delta\|f\|_2^2 \qquad \text{for } f \in \mathcal{F} \text{ with } \|f\|_1 \leq 1,$$

where $\theta(r) = -\varphi'(\varphi^{-1}(r))$ and c_1, c_2 are positive constants.

(ii) *For any $\varepsilon \in (0,1)$ there are constants $C_\varepsilon, c_\epsilon > 0$ so that for all $t > 0$ and $x, y \in E \setminus \mathcal{N}$,*

$$p(t,x,y) \leq C_\varepsilon\, \varphi(c_\varepsilon t) e^{\delta t} \exp\left(-|\psi(y) - \psi(x)| + (1+\varepsilon)\Lambda(\psi)^2 t\right),$$

where

$$\Lambda(\psi)^2 := \max\left\{ \left\| \frac{d e^{-2\psi}\Gamma(e^{\psi}, e^{\psi})}{dm} \right\|_\infty, \left\| \frac{d e^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})}{dm} \right\|_\infty \right\} < \infty.$$

Nash-type inequalities and off-diagonal heat kernel estimates

- $$p(t,x,y) \le C_\varepsilon\, \varphi(c_\varepsilon t) e^{\delta t} \exp\left(-|\psi(y)-\psi(x)| + (1+\varepsilon)\Lambda(\psi)^2 t\right),$$

where

$$\Lambda(\psi)^2 := \max\left\{ \left\| \frac{de^{-2\psi}\Gamma(e^{\psi}, e^{\psi})}{dm} \right\|_\infty, \left\| \frac{de^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})}{dm} \right\|_\infty \right\} < \infty;$$

- $$p(t,x,y) \le C_\varepsilon \varphi(c_\varepsilon t) e^{\delta t} \exp\left(-\frac{d_{\mathcal{E}}(x,y)^2}{4(1+\varepsilon)t}\right),$$

where

$$d_{\mathcal{E}}(x,y) := \sup\left\{\psi(x)-\psi(y) : \ \psi \in \mathcal{F} \cap C_b(E) \text{ with } \Lambda(\psi) \le 1\right\}.$$

Nash-type inequalities and off-diagonal heat kernel estimates

- $$p(t,x,y) \le C_\varepsilon\, \varphi(c_\varepsilon t) e^{\delta t} \exp\left(-|\psi(y)-\psi(x)| + (1+\varepsilon)\Lambda(\psi)^2 t\right),$$

where

$$\Lambda(\psi)^2 := \max\left\{\left\|\frac{de^{-2\psi}\Gamma(e^{\psi},e^{\psi})}{dm}\right\|_\infty, \left\|\frac{de^{2\psi}\Gamma(e^{-\psi},e^{-\psi})}{dm}\right\|_\infty\right\} < \infty;$$

- $$p(t,x,y) \le C_\varepsilon \varphi(c_\varepsilon t) e^{\delta t} \exp\left(-\frac{d_{\mathcal{E}}(x,y)^2}{4(1+\varepsilon)t}\right),$$

where

$$d_{\mathcal{E}}(x,y) := \sup\left\{\psi(x)-\psi(y) : \ \psi \in \mathcal{F}\cap C_b(E) \text{ with } \Lambda(\psi)\le 1\right\}.$$

Nash-type inequalities and off-diagonal heat kernel estimates

- $$\Lambda(\psi)^2 := \max\left\{\left\|\frac{de^{-2\psi}\Gamma(e^{\psi}, e^{\psi})}{dm}\right\|_\infty, \left\|\frac{de^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})}{dm}\right\|_\infty\right\} < \infty.$$

- $$p(t,x,y) \le C_\varepsilon \varphi(c_\varepsilon t) e^{\delta t} \exp\left(-\frac{d_{\mathcal{E}}(x,y)^2}{4(1+\varepsilon)t}\right),$$

where

$$d_{\mathcal{E}}(x,y) := \sup\left\{\psi(x) - \psi(y) : \ \psi \in \mathcal{F} \cap C_b(E) \text{ with } \Lambda(\psi) \le 1\right\}.$$

- Strongly local Dirichlet forms: $d^*_{\mathcal{E}}(x,y)$ is the intrinsic distance induced by the Dirichlet form $(\mathcal{E}, \mathcal{F})$, i.e.,

$$d^*_{\mathcal{E}}(x,y) := \sup\left\{\psi(x) - \psi(y) : \ \psi \in \mathcal{F} \cap C_b(E) \text{ with } d\Gamma(\psi,\psi)/dm \le 1\right\}.$$

Nash-type inequalities and off-diagonal heat kernel estimates

- $$\Lambda(\psi)^2 := \max\left\{\left\|\frac{de^{-2\psi}\Gamma(e^{\psi},e^{\psi})}{dm}\right\|_\infty, \left\|\frac{de^{2\psi}\Gamma(e^{-\psi},e^{-\psi})}{dm}\right\|_\infty\right\} < \infty.$$

- $$p(t,x,y) \le C_\varepsilon \varphi(c_\varepsilon t)e^{\delta t}\exp\left(-\frac{d_{\mathcal{E}}(x,y)^2}{4(1+\varepsilon)t}\right),$$

where

$$d_{\mathcal{E}}(x,y) := \sup\left\{\psi(x)-\psi(y):\ \psi \in \mathcal{F}\cap C_b(E) \text{ with } \Lambda(\psi)\le 1\right\}.$$

- Strongly local Dirichlet forms: $d^*_{\mathcal{E}}(x,y)$ is the intrinsic distance induced by the Dirichlet form $(\mathcal{E},\mathcal{F})$, i.e.,

$$d^*_{\mathcal{E}}(x,y) := \sup\left\{\psi(x)-\psi(y):\ \psi \in \mathcal{F}\cap C_b(E) \text{ with } d\Gamma(\psi,\psi)/dm \le 1\right\}.$$

Remark

- $\varphi(t) = c\,t^{-\nu/2}$ and $\varphi(t) = c\left(t^{-\nu/2}\mathbb{1}_{\{t\le 1\}} + t^{-\mu/2}\mathbb{1}_{\{t>1\}}\right)$ for some constants $c > 0$ and $0 < \mu \le \nu < \infty$. See Carlen-Kusuoka-Stroock (1987).

- $\varphi \in \mathcal{R}$. Typical examples for regular functions on $\mathbb{R}_+$ are $\varphi(r) = r^{-\nu/2}$ with $\nu > 0$, or $\varphi(r) = r^{-\nu/2}$ with $\nu > 0$ for small $r > 0$, and $\varphi(r) = e^{-r^\alpha}$ with $\alpha \in (0, 1]$ for large $r > 0$. Indeed, for any decreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = \infty$, $\varphi(\infty) = 0$ and that $1/\varphi$ has the doubling property (i.e., there is a constant $c_1 \ge 1$ such that $1/\varphi(2r) \le c_1/\varphi(r)$ for all $r > 0$), we can find some $\overline{\varphi} \in \mathcal{R}$ and a constant $c_2 \ge 1$ so that $c_2^{-1}\overline{\varphi}(r) \le \varphi(r) \le c_2\overline{\varphi}(r)$ for all $r \in \mathbb{R}_+$.

Remark

• Carlen-Kusuoka-Stroock (1987): in order to study off-diagonal heat kernel bounds for $(P_t)_{t\geq 0}$, we consider the perturbed semigroup $(P_t^\psi)_{t\geq 0}$ defined by

$$P_t^\psi f(x) := e^{\psi(x)} P_t(e^{-\psi} f)(x), \quad t \geq 0$$

for some nice function $\psi \in \mathcal{F}_b$.

• Carlen-Kusuoka-Stroock (1987): The approach is based on differential inequalities below, which seems to be specific to $\varphi(t)$ of being $c\, t^{-\nu/2}$ considered there:

$$u'(t) \leq -\frac{\varepsilon}{p}\left(\frac{t^{(p-2)/(\beta p)}}{w(t)}\right)^{\beta p} u^{1+\beta p}(t) + \lambda p u(t), \quad t > 0$$

for some increasing function $w(t)$ on $(0, \infty)$ and $p > 2$.

• On-diagonal heat kernel upper bounds for mixtures of symmetric stable-like processes on $\mathbb{R}^d$ are of the form $c_4(\Phi^{-1}(t))^d$ for some strictly increasing weighted function Φ which satisfies doubling and reverse doubling properties; see Chen-Kumagai (2008).

Remark

• Carlen-Kusuoka-Stroock (1987): in order to study off-diagonal heat kernel bounds for $(P_t)_{t\geq 0}$, we consider the perturbed semigroup $(P_t^\psi)_{t\geq 0}$ defined by

$$P_t^\psi f(x) := e^{\psi(x)} P_t(e^{-\psi} f)(x), \quad t \geq 0$$

for some nice function $\psi \in \mathcal{F}_b$.

• Carlen-Kusuoka-Stroock (1987): The approach is based on differential inequalities below, which seems to be specific to $\varphi(t)$ of being $c\,t^{-\nu/2}$ considered there:

$$u'(t) \leq -\frac{\varepsilon}{p}\left(\frac{t^{(p-2)/(\beta p)}}{w(t)}\right)^{\beta p} u^{1+\beta p}(t) + \lambda p u(t), \quad t > 0$$

for some increasing function $w(t)$ on $(0, \infty)$ and $p > 2$.

• On-diagonal heat kernel upper bounds for mixtures of symmetric stable-like processes on $\mathbb{R}^d$ are of the form $c_4(\Phi^{-1}(t))^d$ for some strictly increasing weighted function Φ which satisfies doubling and reverse doubling properties; see Chen-Kumagai (2008).

Applications

• Heat kernel for Brownian motion on a non-compact manifold with bounded geometry; see Barlow-Coulhon-Grigor'yan (01).

• Heat kernel for Brownian motion on hyperbolic spaces and its subordination; see Grigor'yan (94). This gives us a concrete example of symmetric jump process whose heat kernel decays exponentially for large time; see Schilling-W. (02).

• Symmetric Lévy-like processes with general scaling functions; see Chen-Kumagai (08), Mimica (12), Bae-Kang-Kim-Lee (19) and Chen-Kumagai-W. (19).

Questions

- Carlen-Kusuoka-Stroock (1987): *However, it is often important to work with a discrete time parameter; and so in the present section we develop the discrete-time analogs of the results in section 2. Unfortunately, we do not know how to extend the results of section 3 to this setting.*

- Heat kernel estimates for degenerate parabolic equations.

Example 1

let $A(x)$ *be a symmetric measurable matrix-valued function on* $\mathbb{R}^d$ *such that there are some constants* $0 < \lambda_1 \le \lambda_2 < \infty$ *so that*

$$\lambda_1|\xi|^2 \le A(x)\xi\cdot\xi \le \lambda_2|\xi|^2 \quad \text{for every } x,\xi\in\mathbb{R}^d.$$

Let $\mu(dx) = \rho(x)\,dx$ *with* $\rho > 0$ *so that* $\rho+\rho^{-1} \in L^1_{\rm loc}(\mathbb{R}^d;dx)$. *Consider the following regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $L^2(\mathbb{R}^d;\mu)$*:*

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} A(x)\nabla f(x)\cdot\nabla f(x)\,\mu(dx), \quad f\in\mathcal{F}.$$

Questions

• Carlen-Kusuoka-Stroock (1987): *However, it is often important to work with a discrete time parameter; and so in the present section we develop the discrete-time analogs of the results in section 2. Unfortunately, we do not know how to extend the results of section 3 to this setting.*

• Heat kernel estimates for degenerate parabolic equations.

Example 1

let $A(x)$ be a symmetric measurable matrix-valued function on $\mathbb{R}^d$ such that there are some constants $0 < \lambda_1 \le \lambda_2 < \infty$ so that

$$\lambda_1|\xi|^2 \le A(x)\xi \cdot \xi \le \lambda_2|\xi|^2 \quad \textit{for every } x, \xi \in \mathbb{R}^d.$$

Let $\mu(dx) = \rho(x)\,dx$ with $\rho > 0$ so that $\rho + \rho^{-1} \in L^1_{\rm loc}(\mathbb{R}^d; dx)$. Consider the following regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; \mu)$:

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} A(x)\nabla f(x) \cdot \nabla f(x)\, \mu(dx), \quad f \in \mathcal{F}.$$

Off-diagonal heat kernel estimates for large time

• The state space $(\mathcal{X}, \rho, \mu)$ satisfies the VD and RVD conditions. Let ϕ be an increasing function on $[0, \infty)$ with $\phi(0) = 0$ and satisfying the weak scaling property.

• Let $(\mathcal{E}, \mathcal{F})$ be a strongly local Dirichlet form on $L^2(\mathcal{X}; \mu)$ so that

(i) (Faber-Krahn inequality) There exist constants $c_0, \nu > 0$ and $R_0 \geq 0$ such that for any ball $B := B(x_0, R)$ with $x_0 \in \mathcal{X}$ and $R > R_0$ and for any open set $D \subset B$,

$$\lambda_1(D) \geq \frac{c_0}{\phi(R)} \left(\frac{\mu(B)}{\mu(D)} \right)^{\nu}.$$

where $\lambda_1(D)$ is the principal Dirichlet eigenvalue.

(ii) There is a constant $c_* > 0$ such that for any cut-off function $\psi \in \mathcal{F}_b$ for $B(x_0, r) \subset B(x_0, R)$ with $x_0 \in \mathcal{X}$ and $0 < r < R$ μ-a.e. on $\mathcal{X} \setminus B(x, R)$),

$$\frac{d\Gamma(\psi, \psi)}{d\mu} \leq \frac{c_*}{\phi(R - r)}.$$

Off-diagonal heat kernel estimates for large time

Theorem 4 (Chen and W., 21+)

Under the assumptions above, there exists a properly exceptional set $\mathcal{N} \subset \mathcal{X}$, *and the semigroup* $(P_t)_{t\geq 0}$ *associated with* $(\mathcal{E}, \mathcal{F})$ *has a transition density function* $p(t,x,y)$ *that is defined on* $(2T_0, \infty) \times (\mathcal{X} \setminus \mathcal{N}) \times (\mathcal{X} \setminus \mathcal{N})$ *with* $T_0 := \phi(2R_0)$ *such that there is a constant* $c_1 > 0$ *so that for all* $t > 2T_0$ *and* $x, y \in \mathcal{X} \setminus \mathcal{N}$,

$$p(t,x,y) \leq \frac{c_1}{\sqrt{V(x,\phi^{-1}(t))V(y,\phi^{-1}(t))}}\left(1 + \frac{d_{\mathcal{E}}(x,y)^2}{4t}\right)^{(1+\nu)/\nu} \exp\left(-\frac{d_{\mathcal{E}}(x,y)^2}{4t}\right),$$

where

$$d_{\mathcal{E}}(x,y) := \sup\left\{\varphi(x) - \varphi(y) : \ \varphi \in \mathcal{F} \cap C_c(\mathcal{X}) \text{ with } \Lambda(\varphi)^2 \leq 1\right\}$$

and

$$\Lambda(\varphi) := \left\| \frac{d\Gamma(\varphi,\varphi)}{d\mu} \right\|_\infty^{1/2}.$$

Thank you!